Integrability of the square-triangle random tiling model

Jan de Gier and Bernard Nienhuis
Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

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It is shown that the square-triangle random tiling model is equivalent to an asymmetric limit of the three coloring model on the honeycomb lattice. The latter model is known to be the O(n) model at $T=0$ and corresponds to the integrable model connected to the affine $A_2^{(1)}$ Lie algebra. Thus it is shown that the weights of the square-triangle random tiling satisfy the Yang-Baxter equation, albeit in a singular limit of a more general model. The three coloring model for general vertex weights is solved by an algebraic Bethe ansatz. [S1063-651X(97)08303-7]

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I. INTRODUCTION

Random tiling models have gained renewed interest in the past years by the discovery of quasicrystals [1]. They provide an example of the entropic stability of structures whose diffraction pattern has a rotational symmetry which is incompatible with periodicity. As such they offer an explanation of the existence of quasicrystalline alloys [2]. In analogy with the diamond covering used to describe the ground state configurations of the triangular Ising antiferromagnet [3], random tilings can be described by a domain wall structure. The difficulty is that there is more than one type of domain wall, in contrast to the diamond covering. This fact makes it much more difficult to solve the model by coordinate Bethe ansatz, as was tried for the octagonal square-rhombus tiling [4]. Widom [5], however, succeeded in reducing the diagonalization of the transfer matrix to a set of coupled nonlinear equations using the Bethe ansatz for the square-triangle random tiling model. This random tiling can exhibit a twelvefold rotational symmetry if the area fractions of squares and triangles are both equal to 1/2. For example, a two-dimensional binary alloy of Lennard-Jones atoms, whose equilibrium state is a twelvefold quasicrystal, is well approximated by a random tiling of the plane by squares and triangles [6]. Also, high resolution lattice images of twelvefold quasicrystals in NiCr and NiV alloys [7,8] show atomic positions at vertices of tilings containing primarily squares and triangles. We refer the reader to [9] for more background information on the square-triangle random tiling and its physical applications.

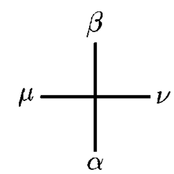
Shortly after the Bethe ansatz solution, Kalugin [10] was able to find a closed expression for the entropy as a function of the domain wall densities in part of the phase diagram using Widoms equations. More recently a solution of an octagonal random tiling has also been found [11].

Being solvable by coordinate Bethe ansatz, it is natural to look for solutions of the Yang-Baxter equation [12] for these models. This would provide an answer to why these models are integrable and a canonical way to diagonalize the transfer matrix via the algebraic Bethe ansatz [13]. The matrix of Boltzmann weights [12], however, is not invertible, which is necessary to obtain commutativity of the transfer matrix. In this paper we show that the square-triangle random tiling model is a singular limit case of a more general model obeying the Yang-Baxter equation.

II. THE MODEL

We shall consider a vertex model on the square lattice, whose Boltzmann weights are denoted by

$$W(\mu,\alpha;\beta,\nu|u) = \tag{1}$$



Each edge of the lattice can be in one of three different states, 1,2, and 3. The partition function of the model is given by

$$Z=\sum_{\text{config.}}\prod_i W(i), \tag{2}$$

where we sum over all configurations which are weighted by the product over the vertices i of their local Boltzmann weights $W(i)$. The explicit form of the weights can be found in Table I. The model can be written in terms of the weights $W_0(\mu,\alpha;\beta,\nu)$ associated with the affine Lie algebra $A_2^{(1)}$, which can be found in [14–16].

$$W(\mu,\alpha;\beta,\nu|u)=O_{\mu\mu'}W_0(\mu',\alpha;\beta,\nu'|u)O_{\nu'\nu}, \tag{3}$$

where $O=\mathrm{diag}\{x_1,x_2,x_3\}$. The weights $W_0(\alpha,\nu;\beta,\nu)$ satisfy the star-triangle or Yang-Baxter equation (YBE) [12],

$$\sum_{\gamma,\mu'',\nu''} W_0(\mu,\nu;\nu'',\mu''|v-u)W_0(\mu'',\alpha;\gamma,\mu'|v)W_0(\nu'',\gamma;\beta,\nu'|u)$$
$$=\sum_{\gamma,\mu'',\nu''} W_0(\nu,\alpha;\gamma,\nu''|u)W_0(\mu,\gamma;\beta,\mu''|v)W_0(\mu'',\nu'';\nu',\mu'|v-u). \tag{4}$$

Equation (4) can be written more elegantly by defining the operators $L(u)$

$$(L(u)_{\mu\nu})_{\alpha\beta}=W_0(\mu,\alpha;\beta,\nu|u),$$

$$L_a(u):\ V_a\otimes\mathbb{C}^3\to V_a\otimes\mathbb{C}^3,\quad V_a\simeq\mathbb{C}^3. \tag{5}$$

The auxiliary label a is introduced for later convenience. We shall omit it if there is no confusion on which space $L(u)$ acts. In this language the vertex states α ($\alpha=1,2,3$) are represented by the standard basis e_α of $\mathbb{C}^3$. $V_a\simeq V_b\simeq\mathbb{C}^3$ are so-called auxiliary spaces, corresponding to the horizontal edges of a vertex. The transfer matrix $\mathcal{T}=\Sigma_{\mu=1}^3 T(u)_{\mu\mu}$ on a lattice of horizontal size N can be written in terms of the local operators $L_a(u)$ (5) as

$$T_a(u)=\left(\prod_{j=1}^{N} O_a^2 L_a^j(u)\right),\quad T_a(u):\ V_a\otimes\mathcal{H}\to V_a\otimes\mathcal{H},$$

$$\mathcal{H}=\bigotimes_{j=1}^{N}\mathbb{C}^3,\quad \mathcal{T}(u)=\mathrm{Tr}_a T_a(u). \tag{6}$$

The trace is taken only over the auxiliary space matrix structure. The operators $L_a^j(u)$ act as $L_a(u)$ on the jth factor in $\mathcal{H}$ and as the identity on all other factors,

$$L^j(u)_{\mu\nu}=\overbrace{I\otimes\cdots\otimes I\otimes}^{\mathrm{j-1\ times}} L(u)_{\mu\nu}\otimes\overbrace{I\otimes\cdots\otimes I}^{\mathrm{N-j\ times}}, \tag{7}$$

where I is the identity on $\mathbb{C}^3$. The partition function (2) on a lattice of size $N\times M$ can then be written as

$$Z=\mathrm{Tr}_{\mathcal{H}}\mathcal{T}(u)^M. \tag{8}$$

Furthermore, we define the R matrix as

$$(R(v-u)_{\mu\nu})_{\alpha\beta}=W_0(\mu,\alpha;\beta,\nu|v-u),$$

$$R_{ab}(v-u):\ V_a\otimes V_b\to V_a\otimes V_b\,. \tag{9}$$

Here too, the Roman labels a and b indicate on which auxiliary space R is acting. Greek labels will be used to indicate matrix elements. The YBE (4) can be written as

$$\begin{aligned}(R(v-u)_{\mu\mu''})_{\nu\nu''}&[L(v)_{\mu''\mu'}L(u)_{\nu''\nu'}]\\&=[L(u)_{\nu\nu''}L(v)_{\mu\mu''}](R(v-u)_{\mu''\mu'})_{\nu''\nu'},\end{aligned} \tag{10}$$

where summation over repeated indices is understood. Each element $L(u)_{\mu\nu}$ of $L(u)$ is an operator acting on $\mathbb{C}^3$. We shall regard $R(u-v)$ as a 9×9 matrix which is given explicitly in Table I. In a compact notation the YBE (10) can be written as an operator equation on the tensor product $V_a\otimes V_b\otimes\mathbb{C}^3$. It then assumes the guise

$$R_{ab}(v-u)[L_a(v)\otimes L_b(u)]=[L_b(u)\otimes L_a(v)]R_{ab}(v-u). \tag{11}$$

From Eq. (11) we obtain the YBE for the matrix $T(u)$ as defined in Eq. (6).

$$R_{ab}(v-u)[T_a(v)\otimes T_b(u)]=[T_b(u)\otimes T_a(v)]R_{ab}(v-u). \tag{12}$$

From expression (8) for the partition sum one sees that the leading term is given by the largest eigenvalue of $\mathcal{T}$. For the sake of completeness we give the explicit diagonalization of the transfer matrix using the algebraic Bethe ansatz in the Appendix. This is just a special case of the trigonometric case of Kulish and Reshetikhin [17].

TABLE I. Boltzmann weights W and R matrix corresponding to the weights W_0. Here we use the abbreviations $s_0=\sinh(\lambda)$, $s_1=\sin(u)$, $s_2=\sinh(u+\lambda)$.

$$W(\mu,\mu;\mu,\mu|u)=x_\mu^2\sinh(u+\lambda),$$

$$W(\mu,\nu;\mu,\nu|u)=x_\mu x_\nu e^{-u\,\mathrm{sgn}(\mu-\nu)}\sinh(\lambda),$$

$$W(1,2;2,1|u)=x_1^2y_3^{-2}\sinh(u),$$

$$W(1,3;3,1|u)=x_1^2y_2^{-2}\sinh(u),$$

$$W(2,3;3,2|u)=x_2^2y_1^{-2}\sinh(u),$$

$$W(2,1;1,2|u)=x_2^2y_3^2\sinh(u),$$

$$W(3,1;1,3|u)=x_3^2y_2^2\sinh(u),$$

$$W(3,2;2,3|u)=x_3^2y_1^2\sinh(u),$$

$$R(u)=\begin{pmatrix} s_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & y_3^{-2}s_1 & 0 & e^u s_0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & y_2^{-2}s_1 & 0 & 0 & 0 & e^u s_0 & 0 & 0\\ 0 & e^{-u}s_0 & 0 & y_3^2 s_1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & s_2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & y_1^{-2}s_1 & 0 & e^u s_0 & 0\\ 0 & 0 & e^{-u}s_0 & 0 & 0 & 0 & y_2^2 s_1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & e^{-u}s_0 & 0 & y_1^2 s_1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_2 \end{pmatrix}$$

III. HONEYCOMB LATTICE

As was already shown by Reshetikhin [18], at a special value of the spectral parameter u, the model defined by the weights W_0 factorizes on the honeycomb lattice. In this section we rederive this result for the model in Table I. Consider the operators

$$OL(u)O\tau:\ \mathbb{C}^3\otimes V\to V\otimes\mathbb{C}^3, \tag{13}$$

where τ is the permutation operator in $\mathbb{C}^3\otimes\mathbb{C}^3$ (recall that $V\simeq\mathbb{C}^3$), $\tau(e_\alpha\otimes e_\beta)=e_\beta\otimes e_\alpha$. The eigenvectors of $OL(u)O\tau$ are given by

$$\overline{e_\gamma}=\frac{1}{\sqrt{1+\mathrm{e}^{-2\lambda}}}(x_\beta y_\gamma e_\beta\otimes e_\alpha - y_\gamma^{-1}x_\alpha \mathrm{e}^{-\lambda}e_\alpha\otimes e_\beta),$$

$$\overline{f_{(\beta\alpha)}}=\frac{1}{\sqrt{1+\mathrm{e}^{2\lambda}}}(y_\gamma x_\beta e_\beta\otimes e_\alpha + y_\gamma^{-1}x_\alpha \mathrm{e}^{\lambda}e_\alpha\otimes e_\beta),$$

$$\overline{f_{(\gamma\gamma)}}=e_\gamma\otimes e_\gamma, \tag{14}$$

with (α,β,γ) a cyclic permutation of (1,2,3). These vectors satisfy the eigenvalue equations

$$OL(u)O\tau\overline{e_\gamma}=-x_\alpha x_\beta \sinh(u-\lambda)\overline{e_\gamma},$$

$$OL(u)O\tau\overline{f_{(\mu\nu)}}=x_\mu x_\nu \sinh(u+\lambda)\overline{f_{(\mu\nu)}}. \tag{15}$$

Thus at $u=-\lambda$ the operator $L\tau$ becomes a projector.

Introducing the dual vectors $\overline{e_\gamma^*}$ with the properties

$$\overline{e_\gamma^*}\cdot\overline{e_{\gamma'}}=x_\alpha x_\beta \delta_{\gamma\gamma'}, \tag{16}$$

we can write $OLO\tau$ at $u=-\lambda$ as

$$OL(-\lambda)O\tau=\sinh(2\lambda)\sum_{\gamma=1}^{3}\overline{e_\gamma e_\gamma^*}. \tag{17}$$

Give the y vertices of the honeycomb lattice the weights $\overline{e_\gamma^*}\cdot(e_\beta\otimes e_\nu)$ and the λ vertices the weights $(e_\mu\otimes e_\alpha)\cdot\overline{e_\gamma}$. Graphically, the matrix element $(L(-\lambda)\tau_{\mu\beta})_{\alpha\nu}$, corresponding to the vertex weight $W(\mu,\alpha;\beta,\nu|-\lambda)$, can be written as a sum over products of two vertices of the honeycomb lattice, see Fig. 1.

It follows that the model on the honeycomb lattice with the vertex weights given in Fig. 2 [where (123) $\rightarrow$ (ABC)] has the same partition function as the model in Table I at $u=-\lambda$ on the square lattice. More precisely,

$$\sinh(\lambda)^{-NM}Z_{NM}^{SQ}(u=-\lambda)=Z_{2NM}^{HC}(\lambda). \tag{18}$$

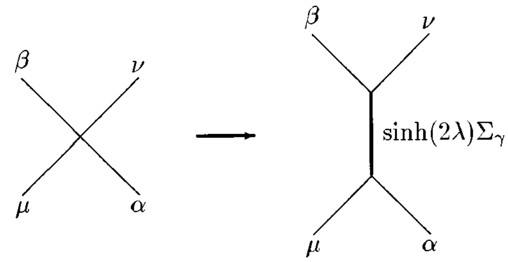


FIG. 1. Factorization of the weights.

The partition function does not change if we apply a gauge transformation to the weights. If we multiply the weights by the factors shown in Fig. 2 and choose $b=-e^{-\lambda/3}a$ and $c=e^{-2\lambda/3}a$ and set $x_i=y_i=1$, we obtain the partition sum of the fully packed loop (FPL) model on the honeycomb lattice [18–21],

$$Z_{NM}^{SQ}(u=-\lambda)=\sinh(\lambda)^{NM}\sum_{\hat{C}} n^{N(\hat{C})}. \tag{19}$$

The sum runs over all dense loop coverings $\hat{C}$ of the honeycomb lattice. $N(\hat{C})$ is the number of loops in the covering $\hat{C}$ and $n=2\cosh(\lambda)$ is the loop fugacity.

The edge states may be interpreted as differences modulo three (going clockwise around each vertex) between three-state Potts variables on the vertices of the triangular lattice. If the states A, B and C are interpreted as differences 0,1, and 2, the corresponding Potts model allows only configurations on the triangle in which two variables are equal and the third is different. The fully ferromagnetic and the completely antiferromagnetic arrangements are then excluded. This model has competing ferromagnetic two-spin and antiferromagnetic three-spin interactions.

IV. SQUARE-TRIANGLE TILING

Take the dual of the honeycomb lattice and associate to each edge of this triangular lattice the corresponding state

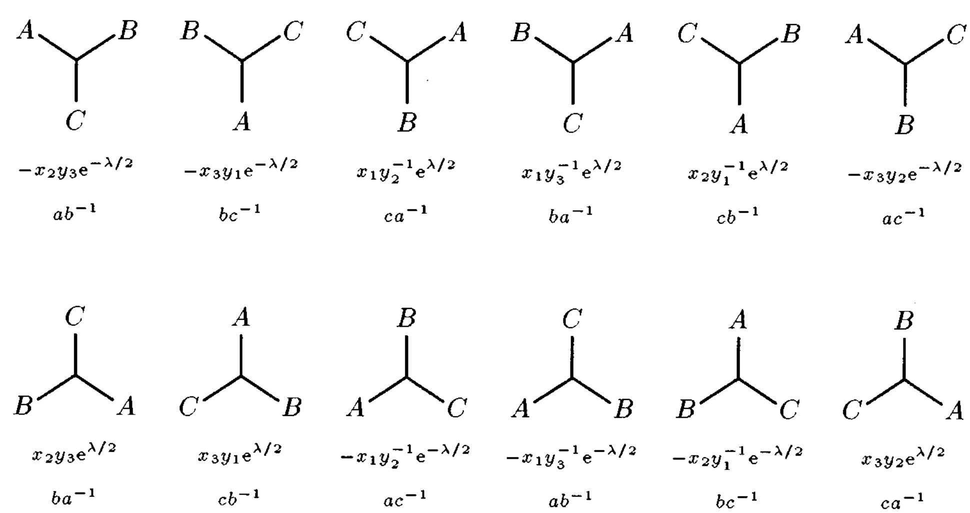


FIG. 2. Vertex configurations on the honeycomb lattice. On the first line their weights and on the second line the gauge transformations are given.

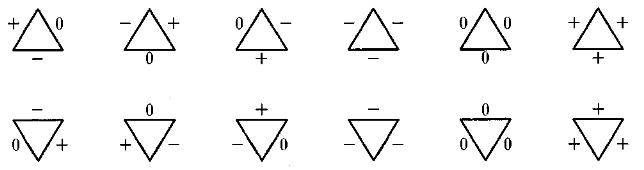


FIG. 3. Face configurations on the triangular lattice corresponding to the vertex configurations in Fig. 2.

variable of the honeycomb lattice. Relabel the states of the horizontal axis by A, B, $C \rightarrow 0, +, -$, on the ascending diagonal by A, B, $C \rightarrow +, -, 0$ and on the descending diagonal by A, B, $C \rightarrow -, 0, +$. The partition sum (18) is then equal to the partition sum of a model on the triangular lattice with face configurations given in Fig. 3.

Now let $x_1^{-1}=x_2=x_3=x^{1/2}$ and $y_1=y_2=y_3=x^{-1/2}$. If $x=0$ the faces with a 0 on all three edges vanish. In that case the states $+$ and $-$ can be regarded as rotation angles of edges with respect to a fixed triangular lattice. Take these angles $\pm\pi/12$ and wipe out every edge with state 0. In this way the model maps onto the square triangle random tiling model. The state 0 corresponds to the diagonal of a square.

The eigenvalue expression (A28) in the limit $u=-\lambda$ becomes

$$\Lambda(-\lambda)=x^{n_1+n_2}\sinh(-\lambda)^N\prod_{k=1}^{n_1}\frac{\sinh(u_k^{(1)})}{\sinh(u_k^{(1)}+\lambda)}\times\prod_{l=1}^{n_2}\frac{\sinh(u_l^{(2)}+2\lambda)}{\sinh(u_l^{(2)}+\lambda)}+x^{n_2}\sinh(-\lambda)^N\prod_{l=1}^{n_2}\frac{\sinh(u_l^{(2)})}{\sinh(u_l^{(2)}+\lambda)}. \tag{20}$$

The Bethe ansatz equations (A26) and (A27), for the two sets of momenta, become

$$\prod_{k=1}^{n_1}\frac{\sinh(u_j^{(2)}-u_k^{(1)})}{\sinh(u_j^{(2)}-u_k^{(1)}+\lambda)}\prod_{l\neq j=1}^{n_2}\frac{\sinh(u_j^{(2)}-u_l^{(2)}+\lambda)}{\sinh(u_j^{(2)}-u_l^{(2)}-\lambda)}=x^{n_1}, \tag{21}$$

and

$$\left(\frac{\sinh(u_j^{(1)}+\lambda)}{\sinh(u_j^{(1)})}\right)^N=x^{N+n_2}\prod_{\substack{k=1\\k\neq j}}^{n_1}\frac{\sinh(u_j^{(1)}-u_k^{(1)}+\lambda)}{\sinh(u_j^{(1)}-u_k^{(1)}-\lambda)}\times\prod_{l=1}^{n_2}\frac{\sinh(u_j^{(1)}-u_l^{(2)}-\lambda)}{\sinh(u_j^{(1)}-u_l^{(2)})}. \tag{22}$$

For $x=1$, these are the Bethe ansatz equations in [18] and [20]. It must be noted that these equations can also be derived from a coordinate Bethe ansatz. For the FPL model this was done by Baxter [22] whose method can be generalized slightly to obtain Eqs. (21) and (22). For our purposes, taking the limit $x\rightarrow 0$, it is more convenient to use Baxter's variables. This can be accomplished by making the following substitutions:

$$s_j=\frac{x\sinh(u_j^{(1)})}{\sinh(u_j^{(1)}+\lambda)},\quad t_\mu=-\frac{x\sinh(u_\mu^{(2)})}{\sinh(u_\mu^{(2)}+\lambda)}. \tag{23}$$

On taking the limit $x\rightarrow 0$ we arrive at the following expressions for the Bethe ansatz equations and the eigenvalue of the transfer matrix of the square-triangle tiling in its triangular lattice representation

$$s_j^N=(-)^{n_1-1}\prod_{\nu=1}^{n_2}(s_j^{-1}-t_\nu^{-1}),\qquad \prod_{k=1}^{n_1}(s_k^{-1}-t_\mu^{-1})=(-)^{n_2-1}, \tag{24}$$

$$\Lambda=\left((-)^{n_2}+\prod_{k=1}^{n_1}s_k\right)\prod_{\nu=1}^{n_2}t_\nu. \tag{25}$$

These equations can be solved analogously to the original solution of the square-triangle random tiling by Kalugin [10].

V. CONCLUSION

We have made a connection between the recently solved square-triangle random tiling model and a known solvable lattice model. It follows from Table I and the substitution $y_i=x^{-1/2}$ that the R matrix for the square-triangle tiling $(x=0)$ is either singular or contains infinite elements. As a result the transfer matrix of the square-triangle tiling model is a limit of a family of commuting transfer matrices, but is itself not a member of such a family. For any finite x though the R matrix is invertible, which implies integrability via the Yang-Baxter equation for this more general model. The square-triangle random tiling thus is a singular limit of a model which is integrable in the usual sense. To obtain the weights of the square-triangle model one has to take the limit $u\rightarrow-\lambda$ first and then take $x\rightarrow 0$. These two limits do not commute.

One final point can be made about the robustness of integrability. The square-triangle tiling has been solved in three different ways. One is that of Widom and Kalugin [5,10], the second can be found in [23] and the third one in this paper. All these methods only differ in their choice of representation, which of course should not influence integrability.

ACKNOWLEDGMENTS

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APPENDIX A: DIAGONALIZATION OF THE TRANSFER MATRIX

We write the matrix $T(u)$ as an operator on $(V^{(0)}\oplus V^{(1)})\otimes\mathcal{H}\simeq V\otimes\mathcal{H}$,

$$T(u)=\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{A1}$$

The entries of $T(u)$ act on the following spaces:

$$A(u)\colon\, V^{(0)}\otimes\mathcal{H}\to V^{(0)}\otimes\mathcal{H},\quad B(u)\colon\, V^{(1)}\otimes\mathcal{H}\to V^{(0)}\otimes\mathcal{H},$$
$$C(u)\colon\, V^{(0)}\otimes\mathcal{H}\to V^{(1)}\otimes\mathcal{H},\quad D(u)\colon\, V^{(1)}\otimes\mathcal{H}\to V^{(1)}\otimes\mathcal{H}. \tag{A2}$$

$V^{(0)}\simeq\mathbb{C}$ and $V^{(1)}\simeq\mathbb{C}^2$ are the subspaces of V corresponding to the natural embedding of $\mathbb{C}^2$ in $\mathbb{C}^3$. The R matrix has the following form on the standard basis of $(V_a^{(0)}\oplus V_a^{(1)})\otimes(V_b^{(0)}\oplus V_b^{(1)})$,

$$R_{ab}(u)=\begin{pmatrix} \sinh(u+\lambda) & 0 & 0 & 0 \\ 0 & \sinh(u)U_b^{-1} & e^{u}\sinh(\lambda)I^{(1)(0)} & 0 \\ 0 & e^{-u}\sinh(\lambda)I^{(0)(1)} & \sinh(u)U_a & 0 \\ 0 & 0 & 0 & R_{ab}^{(1)}(u) \end{pmatrix}. \tag{A3}$$

Here, $V_a^{(0)}\otimes V_b^{(1)}\xrightarrow{I^{(0)(1)}}V_a^{(1)}\otimes V_b^{(0)}\xrightarrow{I^{(1)(0)}}V_a^{(0)}\otimes V_b^{(1)}$ are canonical isomorphisms and

$$U=\begin{pmatrix} y_3^2 & 0 \\ 0 & y_2^2 \end{pmatrix}. \tag{A4}$$

With the notation U_a we denote the operator that acts as U in the space $V_a^{(1)}$ and trivial everywhere else. The reduced matrix $R_{ab}^{(1)}\colon\, V_a^{(1)}\otimes V_b^{(1)}\to V_a^{(1)}\otimes V_b^{(1)}$ has the same structure as the full R matrix

$$R_{ab}^{(1)}(u)=\begin{pmatrix} \sinh(u+\lambda) & 0 & 0 & 0 \\ 0 & y_1^{-2}\sinh(u) & e^{u}\sinh(\lambda) & 0 \\ 0 & e^{-u}\sinh(\lambda) & y_1^{2}\sinh(u) & 0 \\ 0 & 0 & 0 & \sinh(u+\lambda) \end{pmatrix}. \tag{A5}$$

From the relation (12) all commutation relations between the operators A,B,C, and D can be obtained. In the sequel we shall only need the following three of them:

$$A_b(u)\otimes B_a(v)=\sinh(v-u)^{-1}[\sinh(v-u+\lambda)B_a(v)\otimes A_b(u)-e^{v-u}\sinh(\lambda)B_b(u)\otimes A_a(v)I^{(1)(0)}]U_a^{-1}, \tag{A6}$$

$$D_a(v)\otimes B_b(u)=\sinh(v-u)^{-1}U_a^{-1}[B_b(u)\otimes D_a(v)R_{ab}^{(1)}(v-u)-e^{u-v}\sinh(\lambda)I^{(0)(1)}B_a(v)\otimes D_b(u)]. \tag{A7}$$

$$B_a(u)\otimes B_b(v)=\sinh(v-u+\lambda)^{-1}B_b(v)\otimes B_a(u)R_{ab}^{(1)}(v-u). \tag{A8}$$

The first step in diagonalizing $\mathcal{T}$ is to construct candidates for its eigenvectors. This construction will be outlined below. One first defines a ''pseudovacuum'' $F^{(0)}$ on which $T(u)$ is upper trigonal,

$$F^{(0)}=\bigotimes_{j=1}^{N}e_1,\quad e_1=\begin{pmatrix}1\\0\\0\end{pmatrix}. \tag{A9}$$

It then follows that

$$T_a(u)F^{(0)}=\begin{pmatrix} x_1^{2N}\sinh(u+\lambda)^N F^{(0)} & * & * \\ 0 & (x_2^2y_3^2)^N\sinh(u)^N F^{(0)} & * \\ 0 & 0 & (x_3^2y_2^2)^N\sinh(u)^N F^{(0)} \end{pmatrix}. \tag{A10}$$

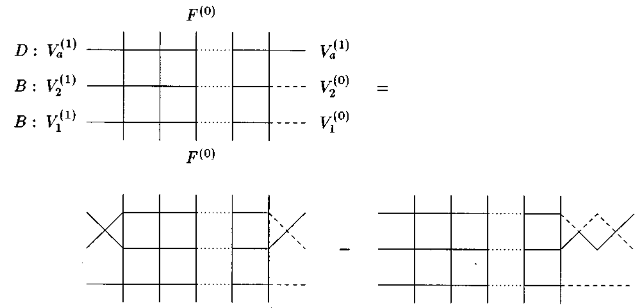

FIG. 4. Commutation rule (A7) for $n_1=2$.

$F^{(0)}$ therefore is an eigenvector of $\mathcal{T}(u)$. To obtain more eigenvectors we make the following ansatz:

$$F=B_{n_1}(u_{n_1}^{(1)})\otimes\cdots\otimes B_1(u_1^{(1)})F^{(1),n_1},$$

$$F^{(1),n_1}\in V_{n_1}^{(1)}\cdots\otimes V_1^{(1)}\otimes F^{(0)}. \tag{A11}$$

The vectors $F^{(1),n_1}$ will be found later. Schematically, we can represent the action of the transfer matrix on F as an $(n_1+1)\times N$ lattice, see Fig. 4. The state $F^{(0)}$ is represented by N vertical edges and the action of each of the $B_i(u_i^{(1)})$ by a horizontal line. The action of $D(u)$ on the resulting vector F is given by the upper horizontal line. The commutation rule (A7) can now be represented graphically by shifting the upper line downwards using the YBE (12). In a similar fashion we can get a graphical representation of Eq. (A6).

The factors that arise after commutation are given in Fig. 4 by the external vertices. For example, the vertex on the left of the second diagram of Fig. 4 corresponds to $R^{(1)}(u-u_1^{(1)})$ and the one on the right to $\sinh(u-u_1^{(1)})^{-1}U_a^{-1}$ It follows from the relations (A6) and (A7) that

$$A(u)F=x_1^{2N}\sinh(u+\lambda)^N\prod_{k=1}^{n_1}\frac{\sinh(u_k^{(1)}-u+\lambda)}{\sinh(u_k^{(1)}-u)}$$
$$\times B_{n_1}(u_{n_1}^{(1)})\otimes\cdots\otimes B_1(u_1^{(1)})U_{n_1}^{-1}\otimes\cdots\otimes U_1^{-1}F^{(1),n_1}$$
$$+(\text{unwanted terms}). \tag{A12}$$

$$D_a(u)F=\prod_{k=1}^{n_1}\sinh(u-u_k^{(1)})^{-1}$$
$$\times B_{n_1}(u_{n_1}^{(1)})\otimes\cdots\otimes B_1(u_1^{(1)})T_a^{(1)}(u;\{u_k^{(1)}\})F^{(1),n_1}$$
$$+(\text{unwanted terms}). \tag{A13}$$

The reduced transfer matrix is given by

$$T_a^{(1)}(u;\{u_k^{(1)}\})=U_a^{-n_1}D_a(u)R_{a1}^{(1)}(u-u_1^{(1)})\cdots R_{an_1}^{(1)}(u-u_{n_1}^{(1)}),$$

$$T_a^{(1)}(u;\{u_k^{(1)}\}):\ \circlearrowleft V_a^{(1)}\otimes V_{n_1}^{(1)}\otimes\cdots V_1^{(1)}\otimes\mathcal{H}. \tag{A14}$$

The "unwanted terms" are similar to Eq. (A12) and (A13) but now one of the B_j has u instead of $u_j^{(1)}$ as its argument. Provided that the "unwanted terms" vanish, the vector F will thus be an eigenvector of $\mathcal{T}(u)=A(u)+\mathrm{Tr}_a D_a(u)$ with eigenvalue $\Lambda(u)$, given by

$$\Lambda(u)=x_1^{2N}\sinh(u+\lambda)^N\prod_{k=1}^{n_1}\frac{\sinh(u_k^{(1)}-u+\lambda)}{\sinh(u_k^{(1)}-u)}\mu(U)$$
$$+\prod_{k=1}^{n_1}\sinh(u-u_k^{(1)})^{-1}\Lambda^{(1)}(u). \tag{A15}$$

$U_{n_1}^{-1}\otimes\cdots\otimes U_1^{-1}$ and $\mathcal{T}^{(1)}(u;\{u_k^{(1)}\})$ are diagonal with eigenvalues $\mu(U)$ and $\Lambda^{(1)}(u)$, respectively. For the "unwanted terms," it is easily seen that the terms in which $B_{n_1}(u_{n_1}^{(1)})$ is replaced by $B_{n_1}(u)$ are (up to a common multiplicative factor) precisely of the form (A12) and (A13) with u and $u_{n_1}^{(1)}$ interchanged and the factor with $k=n_1$ omitted. Using the commutation rule for $B(u)B(v)$ it can be shown that the same also holds for the terms with $B_j(u_j^{(1)})$ replaced by $B_j(u)$. The "unwanted terms" will therefore cancel if $\Lambda(u_j^{(1)})=0$, or

$$x_1^{2N}\mu(U)\sinh(u_j^{(1)}+\lambda)^N\prod_{k=1}^{n_1}\sinh(u_j^{(1)}-u_k^{(1)}-\lambda)$$
$$=-\Lambda^{(1)}(u_j^{(1)}). \tag{A16}$$

The problem of finding the eigenvectors $F^{(1),n_1}$ of $T^{(1)}$ is completely analogous to the construction above. It follows from the commutation rule:

$$R_{ab}^{(1)}(v-u)D_a(v)\otimes D_b(u)=D_b(u)\otimes D_a(v)R_{ab}^{(1)}(v-u) \tag{A17}$$

and the fact that $R_{ab}^{(1)}$ has the same form as R_{ab} that $T^{(1)}$ obeys the YBE

$$R_{ab}^{(1)}(v-u)[T_a^{(1)}(v;\{u_k^{(1)}\})\otimes T_b^{(1)}(u;\{u_k^{(1)}\})]$$
$$=[T_b^{(1)}(u;\{u_k^{(1)}\})\otimes T_a^{(1)}(v;\{u_k^{(1)}\})]R_{ab}^{(1)}(v-u). \tag{A18}$$

Writing $T^{(1)}(u;\{u_k^{(1)}\})$ as

$$T^{(1)}(u;\{u_k^{(1)}\})=\begin{pmatrix} a(u) & b(u)\\ c(u) & d(u)\end{pmatrix}, \tag{A19}$$

since $R^{(1)}(u)$ has the same structure as $R(u)$, we deduce from Eqs. (A6) and (A7) that

$$a(u)b(v)=\sinh(v-u)^{-1}[\sinh(v-u+\lambda)b(v)a(u)$$
$$-e^{v-u}\sinh(\lambda)b(u)a(v)]y_1^{-2}, \tag{A20}$$

$$d(v)b(u)=y_1^{-2}\sinh(v-u)^{-1}[\sinh(v-u+\lambda)b(u)d(v)$$
$$-e^{u-v}\sinh(\lambda)b(v)d(u)], \tag{A21}$$

where now $b(u)$ and $b(v)$ commute. Defining the "pseudovacuum" $F^{(1)(0)}$ in $V_{n_1}^{(1)}\otimes\cdots\otimes V_1^{(1)}$ as

$$F^{(1)(0)}=\bigotimes_{j=1}^{n_1}e_1^{(1)},\quad e_1^{(1)}=\begin{pmatrix}1\\0\end{pmatrix}, \tag{A22}$$

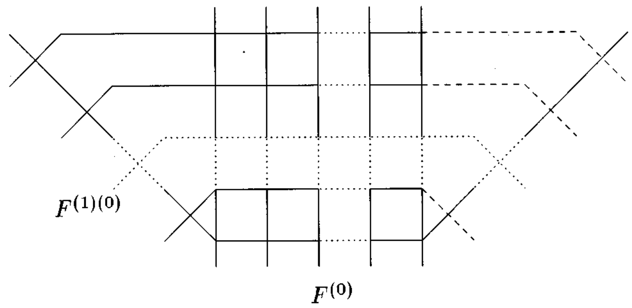

FIG. 5. Action of $\mathcal{T}^{(1)}(u)$ on $F^{(1)(0)}\otimes F^{(0)}$.

the eigenvectors of $\mathcal{T}^{(1)}(u)$ are given by

$$F^{(1),n_1}=b(u_1^{(2)})\cdots b(u_{n_2}^{(2)})F^{(1)(0)}\otimes F^{(0)}. \tag{A23}$$

Graphically, the action of $\mathcal{T}^{(1)}(u)$ on $F^{(1),n_1}$ is depicted in Fig. 5. This diagram arises from Fig. 4 when $D(u)$ is dragged down through all B's and all unwanted diagrams are discarded. The edges on the diagonal on the left in Fig. 5 now represent the ''pseudo-vacuum'' $F^{(1)(0)}$ in the space $V_{n_1}^{(1)}\otimes\cdots\otimes V_1^{(1)}$. The action of the b_i on $F^{(1)(0)}$ is represented by building a lattice on this diagonal in the same way as the operators B_i did on $F^{(0)}$, see Fig. 4. The commutation rules (A20) and (A21) can then be represented graphically similarly.

The eigenvalues $\Lambda^{(1)}$ corresponding to the vectors (A23) are given by

$$\begin{aligned}\Lambda^{(1)}(u)={}&y_3^{-2n_1}(x_2^2y_3^2)^N y_1^{-2n_2}\sinh(u)^N\prod_{k=1}^{n_1}\sinh(u-u_k^{(1)}+\lambda)\\&\times\prod_{l=1}^{n_2}\frac{\sinh(u_l^{(2)}-u+\lambda)}{\sinh(u_l^{(2)}-u)}\\&+y_2^{-2n_1}(x_3^2y_2^2)^N y_1^{2n_1-2n_2}\sinh(u)^N\\&\times\prod_{k=1}^{n_1}\sinh(u-u_k^{(1)})\times\prod_{l=1}^{n_2}\frac{\sinh(u-u_l^{(2)}+\lambda)}{\sinh(u-u_l^{(2)})}.\end{aligned}\tag{A24}$$

The eigenvalue $\mu(U)$ is simply given by

$$\mu(U)=y_3^{-2n_1+2n_2}y_2^{-2n_2}. \tag{A25}$$

The ''unwanted terms'' generated by the action of $T^{(1)}(u)$ on Eq. (A23) can be read off from Eqs. (A20) and (A21). Using the commutativity of $b(u)$ and $b(v)$ they can be shown to cancel and make Eq. (A23) an eigenvector precisely when $\Lambda^{(1)}(u_k^{(2)})=0$. The numbers $\{u_k^{(2)}\}$ therefore satisfy the equations

$$\begin{aligned}&\prod_{k=1}^{n_1}\frac{\sinh(u_j^{(2)}-u_k^{(1)})}{\sinh(u_j^{(2)}-u_k^{(1)}+\lambda)}\prod_{\substack{l=1\\l\neq j}}^{n_2}\frac{\sinh(u_j^{(2)}-u_l^{(2)}+\lambda)}{\sinh(u_j^{(2)}-u_l^{(2)}-\lambda)}\\&\quad=\left(\frac{x_2^2y_3^2}{y_2^2x_3^2}\right)^N\left(\frac{y_2^2}{y_1^2y_3^2}\right)^{n_1}.\end{aligned}\tag{A26}$$

Knowing $\Lambda^{(1)}(u)$ the first set of equations as given by Eq. (A16) becomes

$$\left(\frac{\sinh(u_j^{(1)}+\lambda)}{\sinh(u_j^{(1)})}\right)^N=\left(\frac{x_2^2y_3^2}{x_1^2}\right)^N\left(\frac{y_2^2}{y_1^2y_3^2}\right)^{n_2}\prod_{\substack{k=1\\k\neq j}}^{n_1}\frac{\sinh(u_j^{(1)}-u_k^{(1)}+\lambda)}{\sinh(u_j^{(1)}-u_k^{(1)}-\lambda)}\prod_{l=1}^{n_2}\frac{\sinh(u_j^{(1)}-u_l^{(2)}-\lambda)}{\sinh(u_j^{(1)}-u_l^{(2)})}. \tag{A27}$$

The eigenvalue combining the expressions (A15) and (A24) becomes

$$\begin{aligned}\Lambda(u)={}&x_1^{2N}y_3^{-2n_1+2n_2}y_2^{-2n_2}\sinh(u+\lambda)^N\prod_{k=1}^{n_1}\frac{\sinh(u-u_k^{(1)}-\lambda)}{\sinh(u-u_k^{(1)})}+(x_2^2y_3^2)^N y_3^{-2n_1}y_1^{-2n_2}\sinh(u)^N\prod_{k=1}^{n_1}\frac{\sinh(u-u_k^{(1)}+\lambda)}{\sinh(u-u_k^{(1)})}\\&\times\prod_{l=1}^{n_2}\frac{\sinh(u-u_l^{(2)}-\lambda)}{\sinh(u-u_l^{(2)})}+(x_3^2y_2^2)^N y_2^{-2n_1}y_1^{2n_1-2n_2}\sinh(u)^N\prod_{l=1}^{n_2}\frac{\sinh(u-u_l^{(2)}+\lambda)}{\sinh(u-u_l^{(2)})}.\end{aligned}\tag{A28}$$

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